

Tokamak Transport Based on the Braginskii Model

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Dedicated to Professor Dieter Pfirsch on his 60th Birthday

The two-fluid model of Braginskii is applied to the case of a moderately large tokamak. By estimation of the order of magnitude of the various effects and omission of small terms a somewhat simpler reduced two-fluid Braginskii model is obtained. The model applies on a time scale of order $\tau_e m_i / m_e$, where τ_e is the electron-electron collision time, and energy confinement time is of this order. With electron and ion flow velocities no larger than is necessary to obtain the correct equilibrium currents, classical parallel viscosity becomes a dominant dissipative mechanism. The model allows for the slow evolution of equilibrium states. The equilibria, which include static, ideal magnetohydrodynamic equilibria as a special case, are described. Generally the number density, electrostatic potential, and flows are not constant on a flux surface. The procedure for determination of the slow time evolution of the equilibrium is sketched.

I. Introduction

One of the earliest studies of plasma transport in a toroidal plasma configuration was carried out by Pfirsch and Schlüter [1]. This study is based on the classical fluid model. By taking appropriate averages on a flux surface they obtained a simple expression for particle flow across a flux surface with an enhancement factor of $1 + 2q^2$, where the safety factor $q = 2\pi/l$ is defined as the inverse of the rotational transform. This calculation contained one of the first neoclassical type of transport corrections of order (B/B_p) . Throughout the years Pfirsch has continued to consider transport in plasmas based largely on fluid representations of plasmas, although the fluid representations have often included terms derivable only from kinetic theories [2–6]. In this paper we follow a very similar path. We use a classical two-fluid model of a plasma, but we carry out a systematic expansion in a small parameter in order to obtain the flux surface averaged transport properties implied by the model. In the process we repeat many of the arguments and use the general approach of Pfirsch.

We choose the simplest two-fluid model, which can be relevant to tokamak transport. Specifically

we select the Braginskii two-fluid model, since it is relatively simple and self-contained [7]. A rigorous treatment is applied to this set of general two-fluid equations with flow included. Thus our approach is complementary to a phenomenological model, where anomalous transport coefficients are fitted to match experimental results. Since in the derivation of neoclassical transport coefficients [8] the flow in the basic equilibrium is neglected, the form of transport coefficients commonly used in simulations is questionable. When flows are present relevant quantities such as density, pressure and electrostatic potential vary considerably on a flux surface and this dependence strongly influences transport. We address some of the questions as to the relevance of such a model in the discussion section. Suffice it to say here that we expect that for a plasma state near local thermodynamics equilibrium the Braginskii model allows realistic simulation of a tokamak and contains many of the effects of a more desirable and ultimately essential kinetic model. We believe that the model we have chosen should at least illustrate and give rough quantitative estimates of basic physical phenomena. We tailor our analysis closely to transport in a moderate sized, hot tokamak with parameters $n = 10^{14} \text{ cm}^{-3}$, $B = 4 \text{ T}$, $T_i = T_e = 2.5 \text{ keV}$ and minor radius 50 cm. We then scale all transport coefficients and relevant dimensionless parameters to the small parameter ϵ , where $\epsilon^2 = m_e/m_i$. We then expand the system systematically in ϵ . One essential element of this analysis is that we must have small particle flows in order to generate

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the currents required to form an equilibrium. The large parallel heat conductivity coefficients and the large parallel viscosity coefficients induce major effects on the possible steady states of the system and on the energy transport. In this respect we agree with a recent study based on neoclassical transport coefficients on the importance of the inclusion of parallel viscosity [9]. However, that model omits velocity gradients and does not allow substantial ion flow. The ion mass flow in our model is finite; i.e. at least one order of magnitude larger. A simple-fluid model including flow but neglecting transport has been derived by Zehrfeld and Green [10] and solved numerically later by [11–13]. The poloidal flows necessary to generate a significant poloidal asymmetry are larger by an order of magnitude than the flows in our model. This again indicates that our scaling includes realistic features of a tokamak. We comment in the discussion on the relation between the large parallel viscosity coefficient in the model and the more common neoclassical results.

We find that the expansion in ε gives successive conditions for a steady flow state in successively longer time intervals. We start on the fastest time scale, τ_e , where τ_e is the classical electron-electron collision term. We find a steady state in the τ_e time scale, and with more conditions imposed we may extend the steady state to $\tau_e/\varepsilon \sim \tau_i$. Finally, we are able to extend the lifetime to τ_e/ε^2 , at which point sources are necessary to maintain a steady state or the system evolves on a time scale of order τ_e/ε^2 . In principle it appears possible to maintain a steady state with energy sources only. We find that the usual ideal magnetohydrodynamic equilibria of Grad, Shafranov, and Schlüter are possible, but that more generally the lowest order steady states have electron and ion temperatures constant on flux surfaces, but density varies on a flux surface. The overall determination of the energy dynamics is extremely complicated as it involves higher order perturbations of the steady flow state and the details of our scaling hypotheses. Our analysis is most likely not consistent with the general idea of simple macroscopic scaling laws for tokamak energy confinement times. In the next section we present our scaling assumptions and the reduced Braginskii two-fluid model we study. In Sect. III we look at the lowest order system which characterizes the slowly evolving steady state. Section IV continues with the next order perturbation of the lowest order steady

state and the determination of its time evolution. The final section contains a brief discussion of the use of the Braginskii model and of our results.

II. The Scaling Assumptions and the Reduced Two-Fluid Model

In this section we introduce our scaling hypotheses concerning the many variables appropriate to a two-fluid description of a tokamak. We start with the relatively simple and explicit two-fluid model of Braginskii. We believe that it would be highly desirable to use a kinetic model without reduction to a fluid model, but expect that many relevant phenomena should be describable within the simpler fluid modelling. Our scalings lead to the conclusion that parallel viscosity and the small particle flows necessary to maintain plasma equilibrium currents together play an important role in tokamak dynamics. The remainder of this paper explores the consequences of this approach to the study of tokamaks. In this section we give a qualitative discussion of the effects of viscosity and flow and we also present our reduced, two-fluid, Braginskii plasma model.

In order to present and to justify our scalings we give in Table 1 a set of parameters for a “typical” tokamak with hydrogen ions, $n = 10^{14} \text{ cm}^{-3}$, $B = 4 \text{ T}$, $T_i = T_e = 2.5 \text{ keV}$, minor radius $a = 50 \text{ cm}$, and Coulomb logarithm 15. We assume that the plasma beta is relatively low and in the range 1–5%, and that the safety factor q is on the order of one. Obviously tokamaks vary considerably, but these parameters seem reasonable and somewhat typical. In Table 1, τ is the species collision time, v_{th} is the species thermal speed, and \bar{v} is the species flow velocity necessary to generate a current B/L , or non-dimensionally

$$\bar{v}/c = (\Omega_c/\omega_p) c/(a\omega_p),$$

where Ω_c is the species cyclotron frequency and ω_p the species plasma frequency.

We scale all parameters in terms of ε , where

$$\varepsilon^2 = m_e/m_i,$$

so that in our case $\varepsilon^2 = 1/1836$, $\varepsilon = 2.3 \times 10^{-2}$, and $\varepsilon^{1/2} = .15$. Table 2 presents our hypotheses concerning the parameter scalings. We assume that the

Table 1. Tokamak parameters.

Species \ Parameter	τ	v_{th}	$\Omega_c \tau$	$v_{th} \tau/a$	\bar{v}/v_{th}
e	3×10^{-5} s	2.1×10^9 cm/s	2×10^7	1.3×10^3	1.9×10^{-2}
i	1.8×10^{-3} s	5×10^7 cm/s	6.8×10^5	1.8×10^3	8×10^{-1}

Table 2. Parameter scalings.

Species \ Parameter	Beta = 0 (ϵ). Aspect ratio = 0 ($\epsilon^{-1/2}$)				
	$\Omega \tau$	$v_{th} \tau/a$	\bar{v}/v_{th}	v_t/v_{th}	v_p/v_{th}
e	$\epsilon^{-9/2}$	$\epsilon^{-3/2}$	$\epsilon^{3/2}$	ϵ^2	$\epsilon^{5/2}$
i	$\epsilon^{-7/2}$	$\epsilon^{-3/2}$	$\epsilon^{1/2}$	ϵ	$\epsilon^{3/2}$

plasma β is of order ϵ , and we introduce the major radius R and aspect ratio R/a , which we take to be of order $\epsilon^{-1/2}$. With β of order ϵ , we take B_p/B_t to be of order $\epsilon^{1/2}$, where the subscripts p and t refer to poloidal and toroidal components, respectively. Such scalings of aspect ratio and magnetic field automatically lead to safety factors of order one. In a low beta system we assume that the toroidal current is of order $\epsilon^{1/2}$ times that generated by \bar{v} , while the poloidal current is smaller by an additional factor of $\epsilon^{1/2}$. We assume comparable relationships for the toroidal and poloidal components of flow velocities.

Although we have scaled the aspect ratio in ϵ , we do not, in fact, make an aspect ratio expansion. We carry along in our equations in any order, the toroidal corrections of order $\epsilon^{1/2}$ higher. It is easy to verify that in the systems of equations given the toroidal corrections are larger than the terms omitted, and hence physically relevant and significant.

The scaling hypotheses of Table 2 are moderately reasonable, although a scaling $v_{th} \tau/a \sim \epsilon^{-2}$ would seem more appropriate. Other variations in scalings are equally possible. At the level of the present description it is quite difficult to distinguish different hypotheses. But, for instance with $v_{th} \tau/a \sim \epsilon^{-2}$ and all velocities larger by a factor $\epsilon^{-1/2}$, a different characterization of the slow evolution of tokamaks appears. The system of equations is similar to what we present here, but significant differences occur. Related to another question, if the flow velocities

are substantially larger than our estimates, then the tokamak steady states become quite complicated as the ion and electron flows must approximately cancel each other, in order that an approximately small current may appear. This additional constraint of approximate velocity cancellation greatly complicates the system of equations. Thus, we take the scalings of Table 2 as reasonable and as a set that generates interesting steady states.

In any study of transport one is often asked: ‘‘What is the confinement scaling law?’’. We believe that within the Braginskii two-fluid model this question has almost no meaning. When we impose scaling such as Table 2, we are greatly restricting the class of tokamaks studied, and we may or may not be able to scale from one set of tokamak parameters to another. As we commented above, shifts in the scaling do have quite significant consequences for the representation of tokamak steady states and consequently for confinement scaling. Thus, we view the work presented here as an initial exploration of the exceeding complex field of tokamak dynamics.

Based on the scalings of Table 2, we can now begin to construct our reduced Braginskii two-fluid model. We do not write out explicitly the full form of the Braginskii model; we employ the form as given in the NRL Plasma Formulary of Book, as revised in 1983 [14]. The system has equations of conservation of mass, momentum, and energy for each species, and in addition to the usual pressure and Lorentz forces, it includes frictional forces, thermal forces, and viscous forces. The viscous forces are given in covariant form in, for instance, Kerner and Weitzner [15, 16]. The energy balance equations contain the effects of these forces, as well as ion and electron heating, ion heat flux, and electron heat flux, the latter consisting of the usual thermal gradient terms, plus a frictional heat flux.

If we were to ignore the viscous forces, then we could take the difference of the two momentum

equations and obtain a more or less standard Ohm's law. From this form of Ohm's law we could recover Pfirsch-Schlüter diffusion, which is generated by the frictional force terms. The frictional force terms in the energy equation are of the form

$$E_{Fr} = J_{\perp}^2 / \sigma_{\perp} + J_{\parallel}^2 / \sigma_{\parallel},$$

where

$$\sigma_{\parallel} = 2 \sigma_{\perp} = 2 n e^2 \tau_e / m_e,$$

and with our scaling

$$E_{Fr} = p_e \varepsilon^4 / \tau_e \sim p_i \varepsilon^4 / \tau_e \sim p_i \varepsilon^3 / \tau_i.$$

Thus, the frictional forces generate evolution of the energy on a time scale of order $\tau_e / \varepsilon^4 \sim \tau_i / \varepsilon^3$. As a standard of comparison we note that the ion or electron heating associated with the ion and electron temperature difference is

$$E_{dT} = 3 (m_e / m_i) n k (T_e - T_i) / \tau_e,$$

so that $E_{dT} \sim p_e \varepsilon^2 / \tau_e$ corresponding to a time scale τ_e / ε^2 . Thus, the temperature difference heating occurs on a much faster time scale than the frictional heating. If we were to try to make the frictional heating comparable with the temperature difference heating, we would have to increase the currents and hence the velocities by a factor of $1/\varepsilon$, which would make the velocities much too large. Thus, in our model we will find frictional heating negligible. We see also from this scaling analysis a reasonable time scale for the evolution of the system is τ_e / ε^2 .

We next turn to the contribution of parallel viscosity. If we calculate the force on the ions coming from the poloidal flow, we find

$$F_v^i \sim p_i \tau_i \varepsilon^{3/2} v_{th}^i / a^2 \sim p_i / a \sim p / a$$

while the force on the electrons is

$$F_v^e \sim p_e \tau_e \varepsilon^{5/2} v_{th}^e / a^2 \sim p_e \varepsilon / a \sim \varepsilon p / a.$$

We have calculated the forces based on the poloidal flow; if one calculates the viscous forces based on the toroidal flow and the explicit Braginskii form, see below, one obtains exactly the same estimate since the toroidal flow occurs in the form

$$R[(\mathbf{B} \cdot \nabla)(v_r / R)] / B \quad \text{and} \quad B_p / B \sim \varepsilon^{1/2}.$$

Thus, ion parallel viscosity affects lowest order pressure balance, while electron parallel viscosity appears in first order. The viscous perpendicular

viscosities appear in much higher order. In the energy equations it is easy to see that

$$E_v^i \sim p_i / \tau_i \sim p / \tau_i \sim \varepsilon p / \tau_e$$

while

$$E_v^e \sim \varepsilon^2 p_e / \tau_e \sim \varepsilon^2 p / \tau_e.$$

Thus, ion viscosity affects energy balance on the $\tau_i = \tau_e / \varepsilon$ time scale while electron viscosity affects energy balance on the τ_e / ε^2 time scale. The viscous forces are then significantly larger than the frictional forces, and we examine the effects of the viscous forces.

We are now prepared to formulate our reduced Braginskii model. We assume the scalings of Table 2, and we look for solutions for which we assume that the time dependence is slow and on the time scale $\tau_e / \varepsilon^2 \sim \tau_i / \varepsilon$. Further, if we examine the energy equations and take into account the large parallel heat conductivity, we find easily

$$T_i = T_i(\psi, t) (1 + O(\varepsilon^3)), \quad (1)$$

$$T_e = T_e(\psi, t) (1 + O(\varepsilon^3)). \quad (2)$$

In fact the electron temperature is a flux function to higher order than ε^3 , but ε^3 is adequate for our analysis. Finally, we add momentum and energy sources of order ε^2 times the variables in question. We do not include mass sources, but we allow mass outflow.

If we now write the Braginskii system correct to $O(\varepsilon^2)$ we find easily that with our scaling conservation of mass is

$$\frac{\partial n}{\partial t} + \nabla(n \mathbf{u}_i) = 0, \quad (3)$$

$$\frac{\partial n}{\partial t} + \nabla(n \mathbf{u}_e) = 0, \quad (4)$$

and conservation of momentum is

$$m_i n (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i + \nabla p_i = e n (\mathbf{E} + \mathbf{u}_i \times \mathbf{B} / c) - \nabla \left[\Pi - 3 \frac{\mathbf{B}\mathbf{B}}{B^2} f_i \right] + \mathbf{P}_i, \quad (5)$$

$$\nabla p_e = p_e - e n (\mathbf{E} + \mathbf{u}_e \times \mathbf{B} / c) - \nabla \left[\Pi - 3 \frac{\mathbf{B}\mathbf{B}}{B^2} f_e \right], \quad (6)$$

where \mathbf{P}_i and \mathbf{P}_e are the $O(\varepsilon^2)$ ion and electron momentum sources, the mass flow term is of order $(p_i / a) \varepsilon^{5/2}$, and hence an example of a toroidal

correction retained, and

$$f_\alpha = \eta_0^\alpha \frac{1}{2} \mathbf{B} \cdot \mathbf{W}^\alpha \cdot \mathbf{B} / B^2, \quad \alpha = i, e, \quad (7)$$

where

$$\eta_0^i = 0.96 n k T_i \tau_i, \quad \eta_0^e = 0.73 n k T_e \tau_e, \quad (8)$$

and the strain matrix \mathbf{W} is

$$W_{kl}^\alpha = \frac{\partial u_k^\alpha}{\partial x_l} + \frac{\partial u_l}{\partial x_k} - \frac{2}{3} \delta_{kl} \nabla \cdot \mathbf{u}^\alpha. \quad (9)$$

Conservation of energy, to the order we treat it, is

$$\begin{aligned} & \frac{3}{2} n k \left[\frac{\partial T_i}{\partial t} + (\mathbf{u}_i \cdot \nabla) T_i \right] + p_i \nabla \cdot \mathbf{u}_i \\ &= \nabla \cdot [K_\parallel^i \hat{\mathbf{B}} (\hat{\mathbf{B}} \cdot \nabla) k T_i] + \nabla \cdot [K_\perp^i \hat{\mathbf{B}} \times \nabla k T_i] \quad (10) \\ & - \frac{1}{2} f_i \hat{\mathbf{B}} \cdot \mathbf{W}_i \cdot \hat{\mathbf{B}} + 3 \frac{m_e}{m_i} n k (T_e - T_i) / \tau_e + E_i, \\ & \frac{3}{2} n k \left[\frac{\partial T_e}{\partial t} + (\mathbf{u}_e \cdot \nabla) T_e \right] + p_e \nabla \cdot \mathbf{u}_e \\ &= \nabla \cdot [K_\parallel^e \hat{\mathbf{B}} (\hat{\mathbf{B}} \cdot \nabla) k T_e] + \nabla \cdot [K_\perp^e \hat{\mathbf{B}} \times \nabla k T_e] \\ & - \frac{1}{2} f_e \hat{\mathbf{B}} \cdot \mathbf{W}_e \cdot \hat{\mathbf{B}} - 3 \frac{m_e}{m_i} n k (T_e - T_i) / \tau_e + E_e \\ & + \nabla \cdot [0.71 n k T_e (\mathbf{u}_e \cdot \nabla) \hat{\mathbf{B}}], \quad (11) \end{aligned}$$

where E_i and E_e are the external ion and electron sources, $\hat{\mathbf{B}} = \mathbf{B} / B$,

$$K_\parallel^i = 3.9 n k T_i \tau_i / m_i, \quad K_\parallel^e = 3.2 n k T_e \tau_e / m_e, \quad (12)$$

and

$$\begin{aligned} K_\perp^i &= 2.5 (n k T_i \tau_i / m_i) / (\Omega_i \tau_i), \\ K_\perp^e &= 2.5 (n k T_e \tau_e / m_e) / (\Omega_e \tau_e). \end{aligned} \quad (13)$$

The electromagnetic equations coupled to this system are

$$\nabla \cdot \mathbf{B} = 0, \quad (14)$$

$$\nabla \times \mathbf{B} = 4 \pi e n (\mathbf{u}_i - \mathbf{u}_e) / c, \quad (15)$$

and

$$\nabla \times \mathbf{E} = \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}. \quad (16)$$

In the system (3), (4), (5), (6), (10), (11), conservation of mass, (3), (4) is exact, conservation of momentum, (5), (6) is correct through terms $\varepsilon^2 p / a$,

and conservation of energy (10), (11) is correct through terms $\varepsilon^2 p / \tau_e \sim \varepsilon p / \tau_i$. This system mixes terms of different orders in ε , and in the next sections we shall undertake a systematic expansion of this system. In preparation for such a systematic expansion we can simplify the system slightly and we also restrict it to the special case of axisymmetry as appropriate to a tokamak. We employ cylindrical coordinates r, θ, z with $\hat{\theta}$ being the toroidal direction.

We may reduce the electromagnetic equations (14), (15), (16) of the system by the introduction of flux functions

$$\mathbf{B} = -\hat{r} \psi_{,z} / r + \hat{\theta} \chi / r + \hat{z} \psi_{,r} / r, \quad (17)$$

where

$$\begin{aligned} \chi_{,z} &= -4 \pi (e/c) r n (u_r^i - u_r^e), \\ \chi_{,r} &= 4 \pi (e/c) r n (u_z^i - u_z^e), \end{aligned} \quad (18)$$

and

$$A^* \psi = r (\psi_{,r} / r)_{,r} + \psi_{,zz} = -4 \pi (e/c) r n (u_\parallel^i - u_\parallel^e). \quad (19)$$

A vector potential associated with \mathbf{B} , $\nabla \times \mathbf{A} = \mathbf{B}$ is

$$\mathbf{A} = \hat{\theta} \psi / r - \hat{r} A_{,z} / r + \hat{z} A_{,r} / r, \quad (20)$$

where

$$A^* A = -\chi \quad (21)$$

and $A = 0$ on the boundary of the domain. We may then easily express \mathbf{E} from (16) as

$$\mathbf{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \hat{\theta} V(t) / 2 \pi r, \quad (22)$$

where $V(t)$ is the sloop voltage, assumed $O(\varepsilon^2)$, and $\partial \mathbf{A} / \partial t \sim A \varepsilon^2 / \tau_e$. We may then use the set (17)–(22) instead of (14)–(16). We may also simplify conservation of energy if we integrate (10) and (11) within the flux surface $\psi(r, z) \leq \bar{\psi}$, and we find

$$\begin{aligned} \alpha = i, e, \quad \int_{\psi \leq \bar{\psi}} dV \left\{ \frac{3}{2} n k \left(\frac{\partial T_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla) T_\alpha \right) \right. \\ \left. + p_\alpha \nabla \cdot \mathbf{u}_\alpha + \frac{1}{2} f_\alpha \hat{\mathbf{B}} \cdot \mathbf{W}_\alpha \cdot \hat{\mathbf{B}} - Q_\alpha - E_\alpha \right\} = 0, \end{aligned} \quad (23)$$

where

$$Q_e = -Q_i = 3 (m_e / m_i) n k (T_e - T_i) / \tau_e. \quad (24)$$

We use conservation of energy only in the form (23).

We can also simplify the form for conservation of mass. We observe that given $\partial n / \partial t$ we may intro-

duce a function μ such that

$$\partial n / \partial t + \nabla \cdot (\mu \nabla \psi / |\nabla \psi|) = 0 \quad (25)$$

and then

$$\begin{aligned} n \mathbf{u}_x = & - \hat{\mathbf{r}} \lambda_{x,z} / r + \hat{\theta} \omega_x r n \\ & + \hat{z} \lambda_{x,r} / r + \mu \nabla \psi / |\nabla \psi|. \end{aligned} \quad (26)$$

We may then finally complete the characterization of the magnetic field and, see (18), (19)

$$\chi = \chi_0 + 4 \pi (e/c) (\lambda_i - \lambda_e), \quad (27)$$

$$\Delta^* \psi = -4 \pi (e/c) r^2 n (\omega_i - \omega_e). \quad (28)$$

We add the simple additional explicit forms

$$\begin{aligned} n \mathbf{u}_x \times \mathbf{B} = & \hat{\theta} [(\mathbf{B} \cdot \nabla) \lambda_x] / r - \chi / r^2 \nabla \lambda_x + n \omega_x \nabla \psi \\ & - \hat{\theta} \mu / r |\nabla \psi| + \mu \chi \nabla \psi \times \nabla \theta / |\nabla \psi|, \end{aligned} \quad (29)$$

$$\begin{aligned} \frac{1}{2} \mathbf{B} \cdot \mathbf{W}_x \cdot \mathbf{B} = & \{ \chi (\mathbf{B} \cdot \nabla) \omega_x - \lambda_{x,z} B_\theta^2 / n r^2 \\ & + \frac{1}{3} B^2 (\lambda_{x,r} n_{,z} - \lambda_{x,z} n_{,r}) / r n^2 \} \\ & + \{ -B_r^2 [\lambda_{x,z} / r n]_{,r} + B_r B_z \\ & \cdot \{ [\lambda_{x,r} / r n]_{,r} - [\lambda_{x,z} / r n]_{,z} \} \\ & + B_z^2 [\lambda_{x,r} / r n]_{,z} \} \\ & + \{ B_r (\mathbf{B} \cdot \nabla) (\mu \psi_{,r} / |\nabla \psi|) \\ & + B_z (\mathbf{B} \cdot \nabla) (\mu \psi_{,z} / |\nabla \psi|) \\ & + B_\theta^2 \mu \psi_{,r} / (r |\nabla \psi|) \\ & + \frac{1}{3} (1/n) B^2 \partial n / \partial t \}. \end{aligned} \quad (30)$$

With (29) the $\hat{\theta}$ component of momentum conservation reduces to

$$\begin{aligned} \frac{m_i}{r} [\lambda_{i,r} (r^2 \omega_i)_{,z} - \lambda_{i,z} (r^2 \omega_i)_{,r}] + e n [V(t) / 2 \pi + \psi_{,t} / c] \\ + \frac{e \mu}{c} |\nabla \psi| - \hat{\theta} r \mathbf{P}_i = \mathbf{B} \cdot \nabla (e/c) \lambda_i + 3 r B_\theta f_i / B^2, \end{aligned} \quad (31)$$

$$\begin{aligned} - e n [V(t) / 2 \pi + \psi_{,t} / c] - \frac{e \mu}{c} |\nabla \psi| - \hat{\theta} r \mathbf{P}_e \\ = \mathbf{B} \cdot \nabla \left(- \frac{e}{c} \lambda_e + 3 r B_\theta f_e / B^2 \right). \end{aligned} \quad (32)$$

We have written (31) and (32) such that all the terms on the left hand side are $O(\varepsilon^2)$. Exact con-

sequences of (31) and (32) are

$$\begin{aligned} \int_{\psi \leq \bar{\psi}} dV \left\{ \frac{m_i}{r} [\lambda_{i,r} (r^2 \omega_i)_{,z} - \lambda_{i,z} (r^2 \omega_i)_{,r}] \right. \\ \left. + e n [V(t) / 2 \pi + \psi_{,t} / c] \right. \\ \left. + \frac{e \mu}{c} |\nabla \psi| - \hat{\theta} \cdot \mathbf{P}_i r \right\} = 0, \end{aligned} \quad (33)$$

$$\begin{aligned} \int_{\psi \leq \bar{\psi}} dV \left\{ e n [V(t) / 2 \pi + \psi_{,t} / c] + \frac{e \mu}{c} |\nabla \psi| \right. \\ \left. + \hat{\theta} \cdot \mathbf{P}_e r \right\} = 0 \end{aligned} \quad (34)$$

so that

$$\begin{aligned} \int_{\psi \leq \bar{\psi}} dV \frac{m_i}{r} [\lambda_{i,r} (r^2 \omega_i)_{,z} - \lambda_{i,z} (r^2 \omega_i)_{,r}] \\ = \int \hat{\theta} \cdot (\mathbf{P}_i + \mathbf{P}_e) r dV. \end{aligned} \quad (35)$$

From (35) it is clear that momentum sources are needed to maintain an ion flow.

We summarize the model we use. We take conservation of mass in the form (25), (26), conservation of momentum in the form (31), (32) plus the r and z components of (5), (6), conservation of energy in the integral form (23), and the electromagnetic equations in the form (17), (27), (28) for \mathbf{B} and (20), (21) and (22) for \mathbf{E} .

III. The Lowest Order Expansion

As was noted in the preceding section, the reduced Braginskii model contains terms of different order in ε and is extremely complex. It thus seems useful to apply a consistent expansion in ε to the system. In this section we study the system to lowest order in ε . This study constitutes the major part of this work and is directed to the characterization of physically and mathematically well-posed problems for the system. In the next section we sketch the analysis to the next order in ε and we give equations for the time evolution of the system in the τ_e / ε^2 time scale.

In our system to lowest order in ε all external source terms, plus all time derivatives, are $O(\varepsilon^2)$ and hence ignored. Furthermore the contribution of electron viscosity in momentum balance is $O(\varepsilon)$, that is $f_e = O(\varepsilon p)$, and hence is ignored. Finally, the

many terms in the expression for strain, see (30), proportional to B_r^2 , $B_r B_z$, and B_z^2 are all $O(\beta) = O(\varepsilon)$ times leading order terms, and hence dropped. We may thus finally write the lowest order model as:

Conservation of mass:

$$n \mathbf{u}_\alpha = -\hat{r} \lambda_{\alpha,z}/r + \hat{\theta} r \omega_\alpha n + \hat{z} \lambda_{\alpha,r}/r, \quad \alpha = i, e. \quad (36)$$

Conservation of momentum:

$$(e/c) \lambda_i + 3r^2 f_i/\chi_0 = F_i(\psi), \quad (37)$$

$$-(e/c) \lambda_e = F_e(\psi), \quad (38)$$

$$\begin{aligned} \nabla p_i = & -en \nabla \varphi - \frac{e \chi_0}{c r^2} \nabla \lambda_i \\ & + \frac{e}{c} n \omega_i \nabla \psi - \nabla f_i - 3 \hat{r} f_i/r, \end{aligned} \quad (40)$$

$$\nabla p_e = en \nabla \varphi + \frac{e \chi_0}{c r^2} \nabla \lambda_e - \frac{e}{c} n \omega_e \nabla \psi, \quad (41)$$

$$\begin{aligned} \frac{1}{2} \mathbf{B} \cdot \mathbf{W}_\alpha \cdot \mathbf{B} = & \chi_0 (\mathbf{B} \cdot \nabla) \omega_\alpha \\ & + \chi_0^2 (\lambda_{\alpha,r} n_{\alpha,z} - \lambda_{\alpha,z} n_{\alpha,r})/3 r^3 n^2 \\ & - \lambda_{\alpha,z} \chi_0^2/(r^4 n), \\ f_\alpha = & \eta_0^\alpha \frac{1}{2} \mathbf{B} \cdot \mathbf{W}_\alpha \mathbf{B}/B^2, \quad \alpha = i, e, \end{aligned} \quad (42)$$

and conservation of energy:

$$T_i = T_i(\psi); \quad T_e = T_e(\psi), \quad (43)$$

$$\int_{\psi \leq \bar{\psi}} \left\{ \frac{3}{2} n k (\mathbf{u}_i \cdot \nabla) T_i + p_i \nabla \cdot \mathbf{u}_i + \frac{1}{2} f_i \mathbf{B} \cdot \mathbf{W}_i \cdot \mathbf{B}/B^2 \right\} dV = 0, \quad (44)$$

$$\int_{\psi \leq \bar{\psi}} \left\{ \frac{3}{2} n k (\mathbf{u}_e \cdot \nabla) T_e + p_e \nabla \cdot \mathbf{u}_e \right\} dV = 0. \quad (45)$$

We note that (44) and (45) represent energy balance on the $\tau_e/\varepsilon = \tau_i$ time scale. Maxwell's equations are just

$$\mathbf{B} = -\hat{r} \psi_z/r + \hat{\theta} \chi/r + \hat{z} \psi_r/r, \quad (46)$$

$$\chi = \chi_0 + O(\varepsilon), \quad (47)$$

$$\Delta^* \psi = -4\pi (e/c) r^2 n (\omega_i - \omega_e), \quad (48)$$

$$E = -\nabla \varphi. \quad (49)$$

Our lowest order model consists of the definitions and equations (36)–(49).

We now start the analysis of this system with an examination of the electron equations. From (38), λ_e is clearly a function of ψ , and hence momentum balance reduces to

$$\nabla p_e = en \nabla \varphi + \frac{e}{c} \left(\frac{\chi_0}{r^2} \lambda'_e(\psi) - n \omega_e \right) \nabla \psi. \quad (50)$$

Thus

$$\frac{1}{n} (\mathbf{B} \cdot \nabla) p_e = \mathbf{B} \cdot \nabla (k T_e \ln n) = e \mathbf{B} \cdot \nabla \varphi$$

or

$$n = N(\psi) \exp(e \varphi/k T_e), \quad (51)$$

and

$$p_e = P_e(\psi) \exp(e \varphi/k T_e). \quad (52)$$

If we insert (51) or (52) in (50) we find

$$\frac{P'_e(\psi)}{N(\psi)} = e \varphi \frac{T'_e(\psi)}{T_e(\psi)} + \frac{e}{c} \left(\frac{\chi_0 \lambda'_e(\psi)}{r^2 n} - \omega_e \right). \quad (53)$$

In addition we note that

$$n \mathbf{u}_e = \lambda'_e(\psi) \mathbf{B} + (n r^2 \omega_e - \chi \lambda'_e) \nabla \theta \quad (54)$$

$$p_e \nabla \cdot \mathbf{u}_e = -k T_e \mathbf{u}_e \cdot \nabla n = -k T_e \lambda'_e \mathbf{B} \cdot \nabla \ln n, \quad (55)$$

so that (45) is trivially satisfied. Hence, in lowest order the electron equations reduce to the relation (51) (or 52) between n (or p_e) and φ and the relation (53) for ω_e . It is useful to carry out one subsidiary calculation before we turn to the ions. From (53) we see that

$$\begin{aligned} \mathbf{B} \cdot \nabla \omega_e + \chi_0 (\lambda_{e,r} n_{e,z} - \lambda_{e,z} n_{e,r})/r^3 n^2 - 2 \chi_0 \lambda_{e,z}/r^4 n \\ = \frac{c}{e} k T'_e \mathbf{B} \cdot \nabla \ln n, \end{aligned}$$

so that, see (41),

$$\begin{aligned} \frac{1}{2} \mathbf{B} \cdot \mathbf{W}_e \cdot \mathbf{B} = & -\frac{2}{3} \lambda'_e(\psi) \chi_0 [\mathbf{B} \cdot \nabla n]/r^2 n^2 \\ & - B_r \chi_0 \lambda'_e(\psi)/r^3 n \\ & + \frac{c}{e} k T'_e \mathbf{B} \cdot \nabla \ln n. \end{aligned} \quad (56)$$

We shall be able to obtain a final system that does not contain φ explicitly.

We next return to the ions. When we insert (37) into momentum balance we find easily

$$\begin{aligned} \nabla p_i = & -en \nabla \varphi - \frac{2}{3} \frac{e \chi_0}{c r^2} \nabla \lambda_i \\ & + \left(\frac{e}{c} n \omega_i - \frac{\chi_0 F'_i(\psi)}{3 r^2} \right) \nabla \psi \\ & + \hat{r} \left(\frac{e}{3c} \frac{\chi_0}{r^3} \lambda_i - \frac{\chi_0 F'_i}{3 r^3} \right). \end{aligned} \quad (57)$$

We can simplify the system somewhat if we add the electron and ion momentum equations and introduce total plasma variables

$$\begin{aligned} T(\psi) &= T_i(\psi) + T_e(\psi), \\ F(\psi) &= F_i(\psi) + F_e(\psi), \\ \lambda &= \lambda_i - \lambda_e(\psi), \\ \omega &= \omega_i - \omega_e, \end{aligned} \quad (58)$$

so that

$$\begin{aligned} \nabla p = & -\frac{2}{3} \frac{e \chi_0}{c r^2} \nabla \lambda + \left(\frac{e}{c} n \omega - \frac{\chi_0 F'(\psi)}{3 r^2} \right) \nabla \psi \\ & - \hat{r} \frac{\chi_0}{3 r^3} \left(\frac{e}{c} \lambda - F(\psi) \right) \end{aligned} \quad (59)$$

and from (41), (42), and (43)

$$\begin{aligned} \mathbf{B} \cdot \nabla \omega + \chi_0 (\lambda_{,r} n_{,z} - \lambda_{,z} n_{,r}) / 3 r^3 n^2 - \lambda_{,z} \chi_0 / r^4 n \\ = \frac{\chi_0}{3 r^4 n_0^1} (F(\psi) - \frac{e}{c} \lambda) \\ + \mathbf{B} \cdot \nabla n \left\{ \frac{2}{3} \frac{\lambda'_e(\psi) \chi_0}{r^2 n^2} - \frac{c}{en} k T'_e \right\} + \frac{B_r \chi_0 \lambda'_e(\psi)}{r^3 n}. \end{aligned} \quad (60)$$

Finally, the Grad-Shafranov-Schlüter equation is just

$$\Delta^* \psi = - \left(\frac{e}{c} \right) r^2 n \omega. \quad (61)$$

Thus, our lowest order steady states are characterized (59), (60), and (61). The electron flow can then be found from (51), (52), and (53), and the ion flow from (58). Although we do not yet know what data are appropriate for (59) and (60), we see that so far there are four arbitrary flux functions that appear: $T_1(\psi)$, $T_e(\psi)$, $F(\psi)$ and $\lambda_e(\psi)$.

Before we continue our examination of (59), (60), (61) it is useful to observe that static ideal MHD equilibria are possible solutions of our system. It is easier to show this directly from the original system (36)–(49). We take $F_1 = 0$, $\lambda_i = 0$, $\varphi = \varphi(\psi)$, $n = n(\psi)$, $\omega_i = \omega_i(\psi)$ and we find

$$\frac{e}{c} n \omega = p'(\psi) - \frac{e \chi_0}{c r^2} \lambda'_e(\omega) \quad (62)$$

so that (61) becomes exactly the Grad-Shafranov-Schlüter equation and

$$p'_i(\psi) = -en \varphi'(\psi) + \frac{e}{c} n \omega_i. \quad (63)$$

We now return to the system (59), (60), (61). We see that (61) essentially decouples from (59), (60) in that with $n\omega$ given we may solve for ψ directly from (61). We might even hypothesize an iteration scheme in which we solve (59) and (60) for n , $n\omega$, and λ , assuming \mathbf{B} given, and then we solve for \mathbf{B} from (61). The structure of (61) as an elliptic equation for which boundary data in ψ are given is quite clear, and we concentrate on (59) and (60), assuming \mathbf{B} as given. The system (59), (60) is far from standard and we must modify it considerably before we can proceed. We observe that (59), (60) is a set of first order partial differential equations for n , $n\omega$, and λ , but the system is not quasi-linear as the terms $\lambda_{,r} n_{,z} - \lambda_{,z} n_{,r}$ appear in (60). We may rewrite (59) as

$$\begin{aligned} k T \nabla n = & -\frac{2}{3} \frac{e \chi_0}{c r^2} \nabla \lambda \\ & + \left(\frac{e}{c} n \omega - \frac{\chi_0 F'(\psi)}{3 r^2} - n k T'(\psi) \right) \nabla \psi \\ & + \hat{r} \frac{\chi_0}{3 r^3} \left(\frac{e}{c} \lambda - F(\psi) \right) \end{aligned} \quad (64)$$

and we may use (64) to eliminate ∇n from $\hat{\theta} \cdot \nabla \lambda \times \nabla n$, and we find

$$\begin{aligned} \mathbf{B} \cdot \nabla \omega - \frac{\chi_0}{3 k T r^2 n^2} \mathbf{B} \cdot \nabla \lambda \\ \cdot \left\{ \frac{e}{c} n \omega - \frac{\chi_0 F'(\psi)}{3 r^2} - n k T' \right\} \\ - \frac{\lambda_{,z} \chi_0}{r^4 n} \left\{ 1 + \frac{\chi_0 (e/c \lambda - F)}{9 r^2 n k T} \right\} \\ - \mathbf{B} \cdot \nabla n \left\{ \frac{2}{3} \frac{\lambda'_e(\psi) \chi_0}{r^2 n^2} - \frac{c}{e} \frac{k T'_e}{n} \right\} \end{aligned}$$

$$= \frac{\chi_0}{3 r^4 \eta_0^{\frac{1}{2}}} \left(F(\psi) - \frac{e}{c} \lambda \right) + \frac{B_r \chi_0 \lambda'_e(\psi)}{r^3 n}. \quad (65)$$

Although (65) is more complicated than the equivalent form (60), the former has the advantage of being a quasilinear equation in ω , λ and n . We then take our basic system as (59) and (65).

Now (59), or (64) has problems in that it is not a standard equation and, in fact, involves integrability constraints. We may obtain a more standard system from (59) by the following procedure. We denote the r and z components of (59) as

$$S_r = 0, \quad S_z = 0, \quad (66), (67)$$

and we wish to construct an equivalent, but more obvious system. Clearly, consequences of (66), (67) are

$$S_{r,z} - S_{z,r} = 0, \quad B_r S_r + B_z S_z = 0. \quad (68), (69)$$

We must determine under what additional conditions, if any, are (68), (69) equivalent to (66), (67). We see easily that (68) is equivalent to

$$(S_r + B_z S_z / B_r)_{,z} - [B \cdot \nabla S_z] / B_r - S_z (B_z / B_r)_{,z} = 0 \quad (70)$$

or to

$$[B \cdot \nabla S_r] / B_z + (B_r / B_z)_{,r} S_r - \left(S_z + \frac{B_r}{B_z} S_r \right)_{,r} = 0. \quad (71)$$

Thus, suppose we consider a tokamak, see Fig. 1,

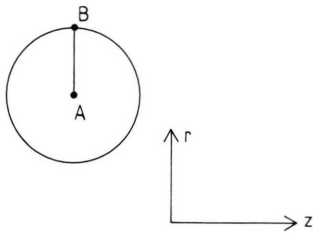


Fig. 1

and in the segment AB from this magnetic axis to edge we give $S_r = 0$ and we assume that (68) and (69) hold everywhere. From (71) we conclude that $S_r \equiv 0$ up to a curve on which $B_z = 0$. Furthermore from (69) we infer that $S_z \equiv 0$ up to that curve also. Thus, see Fig. 2, we infer $S_r = S_z = 0$ in a sector shown $C'AC$.

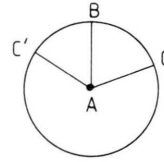


Fig. 2

On $C'AC$, $S_z = 0$ and we then apply (70), so that by the same argument we get $S_r = S_z = 0$ in a sector $D'AD$ which contains AC' , AB , AC .

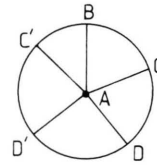


Fig. 3

A final application of (71) then gives $S_r \equiv S_z \equiv 0$ everywhere. Thus, the system (66), (67) is equivalent to the system (68), (69), together with the condition that on $z = 0$ from the magnetic axis to the edge $S_r = 0$.

When we apply the preceding analysis to (59) we conclude that (59) is equivalent to

$$0 = -\frac{e \chi_0}{c r^3} \lambda_{,z} + r B \cdot \nabla \left(\frac{e}{c} n \omega \right) + \frac{B_r \chi_0 F'(\psi)}{r^2}, \quad (72)$$

$$k T B \cdot \nabla n = -\frac{2}{3} \frac{e \chi_0}{c r^2} B \cdot \nabla \lambda + \frac{B_r \chi_0}{3 r^3} \left(\frac{e}{c} \lambda - F(\psi) \right), \quad (73)$$

and the constraint

$$p_{,r} = -\frac{2 e \chi_0}{3 c r^2} \lambda_{,r} + \left(\frac{e}{c} n \omega - \frac{\chi_0 F'(\psi)}{3 r^2} \right) \psi_{,r} + \frac{\chi_0}{3 r^3} \left(\frac{e}{c} \lambda - F(\psi) \right) \quad (74)$$

holds on the curve $z = 0$ from the magnetic axis to the outer edge. Hence, we have reduced the conditions for momentum balance to the three first

order quasilinear equations (65), (72), and (73), together with the constraint on the solution (74) on a particular curve. We recall that in this analysis we take \mathbf{B} as given.

A complete analysis of (65), (72), (73), and (74) is beyond our ability, but we can give some qualitative, plausibility arguments to hypothesize what might be a reasonable, well-posed problem for the system. As a guide to the nature of the problem, we observe that for the original system in the form (41), (42), if $f_i \neq 0$ then solutions of definite parity in z , corresponding to up-down symmetric solutions, do not exist. For, if λ_i, ω_i, n and ψ are all even functions of z , an up-down symmetric state, then $\mathbf{B} \cdot \mathbf{W}_i \cdot \mathbf{B}$ is odd in z while f_i , given by (37), is even, and (42) fails. As an aside, we conclude that on physical grounds f_i should not be too large. In the general, non-symmetric case we find that the construction of single-valued solutions of (65), (72), (73) and (74) is intricate, but straight-forward, when viewed correctly. Symmetry greatly simplifies the problem, but in general we lack symmetry.

If we consider the system (65), (72), (73), and (74) we observe that we have three quasilinear partial differential equations, whose characteristic surfaces $v(r, z) = \text{const}$ satisfy

$$[\mathbf{B} \cdot \nabla v]^2 \left[v_{,z} \chi_0 \left(F(\psi) - \frac{e}{c} \lambda \right) + \mathbf{B} \cdot \nabla v \left\{ 3 \left[\frac{e}{c} \omega + k (T'_i - T'_e) \right] r^4 n + (F'(\psi) + 4 (e/c) \lambda'_e(\psi)) r^2 \right\} \right] = 0. \tag{75}$$

We see that the characteristics are the flux surfaces counted two times plus the surfaces corresponding to the vanishing of the second factor. Several very different possibilities appear depending on the nature of the second type of characteristics. The simplest case, and finally the one that we analyze is the one in which these characteristics are simple curves that enter and leave the region in which we construct an equilibrium, see Figure 4.

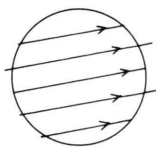


Fig. 4

If we look at the magnetic axis, where $B_p = 0$, we conclude from (65) that $\lambda_{,z}$ and $F(\psi) - e/c \lambda$ are either both zero or both non-zero. We then infer that for these characteristics the magnetic axis is either a center, with closed characteristics nearby, or a node with all characteristics entering and leaving the magnetic axis. Other critical points of these characteristics are also possible. The different types of characteristics do indeed affect the physical properties of the lowest order steady state, but we concentrate on the simplest case given by Figure 4. One additional property of this system is that if we denote the first order differential operator in (75) which generate the second class of characteristics by $L, Lv = 0$, then λ satisfies the equation

$$L \lambda = g(\lambda, n, \omega, r, \mathbf{B}), \tag{76}$$

so that λ alone is associated with those characteristics.

We can now address the question “what is a possible well-posed problem for our system?”. We start with a simpler question, we ask “what is a well-posed problem in the domain $DABCD$, where A is the magnetic axis and DAB is $z = 0$?”, see Figure 5. In Fig. 5 we also indicate by dashed lines

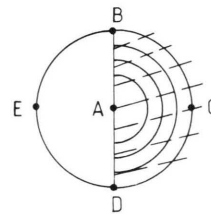


Fig. 5

some of the characteristics of the two families. We can consider this problem as a mixed initial-value boundary-value where we give initial data on AB and boundary data on AD , see Courant Hilbert, Vol. II, p. 471 etc. On AB we must give three pieces of initial data, while on AD , when one characteristic enters the domain, we give one piece of data, which considering (76) should be λ . The general theory does not apply as we have the magnetic axis, a singularity at A , and we want solutions in the large, but qualitatively it seems that we have a plausible characterization of a well-posed problem. We have indicated that the solution depends on four functions: three functions on AB and one function on AD .

However, we recall that the data on AB must satisfy the constraint (74). Hence, we conclude that three arbitrary functions determine a solution of our system in the domain $DABCD$. We have answered the second question, we can now return to the first. The data on DAB would equally well determine a solution in $DABED$. In the symmetric case $f_i \equiv 0$ these solutions would coincide on DAB , but in the non-symmetric case we have no guarantee that the solution in $DABCD$ agrees with the solution $DABED$ on the segment DA . By construction these two solutions have the same value of λ on DA . Thus in order to construct an acceptable solution to our system we must impose the constraints that the two methods of computing n and ω on DA agree. If we impose these two constraints then it is easy to see that n , ω , λ and their derivatives are continuous across DA . Since our solution depended on three free functions and we must impose two constraints we conclude the solution of the system (65), (72), (73), and (74), and hence (59), (60) is expected to have unique well-posed solutions when one free function is specified.

Finally, we expect the lowest order equilibrium system to have unique, well-posed solutions when the four functions $T_i(\psi)$, $T_e(\psi)$, $F_i(\psi)$, $e/c \lambda_e(\psi)$ are given, when one function in the electron equation is given, at (51) and (53), either $N(\psi)$ or the flux surface average of $\omega_e(\psi)$, and when one function in the ion equations is given. Thus, the solution depends on six free functions, and the boundary value of ψ . When we impose the integral constraints in energy balance (44), (45), we conclude that solutions on the τ_i time scale exist and depend on five arbitrary flux functions. In the next section we examine the system in higher order.

IV. The First Order Expansion and the Time Evolution of the System

In this section we discuss the expansion to next order and we examine the slow evolution in time of the steady state described in the previous section. We treat these topics even more briefly and speculatively than we consider the corresponding topics in the previous section. We wish to show that it is a reasonable hypothesis that in next order the system of equations possess solutions and that from these

solutions one can determine the slow evolution in time of the steady state of the preceding section. For the most part, our arguments reduce to the qualitative description of the next order system of partial differential equations together with the enumeration of data necessary to specify solutions. Finally we apply the integral constraints coming from conservation of energy and momentum in order to obtain the time evolution of the system.

We return to our original system of equations: conservation of mass (25), (26), conservation of momentum (31), (32), (5), (6), conservation of energy (23), and the system of electromagnetic equations for \mathbf{B} , (17), (27), (28) and for \mathbf{E} (20), (21), (22). We expand to one higher order in ε , and since $\partial/\partial t \sim \varepsilon^2/\tau_e$, and sources are also $O(\varepsilon^2)$ we find that the system of first order equations is essentially the same as (36)–(49), our lowest order model, with inhomogeneous terms involving only lowest order flows added to many of the equations. Conservation of mass is essentially unchanged,

$$(n\mathbf{u}_x)^{(1)} = -\hat{r} \lambda_{x,r}^{(1)}/r + \hat{\theta} r (n\omega_x)^{(1)} + \hat{z} \lambda_{x,r}^{(1)}/r, \quad (77)$$

where the superscript (1) means the first order terms. Conservation of momentum involves the integration of (35), (36) and the \hat{r} and \hat{z} components of (5) and (6). The sources on the left hand sides of (35), (36) are order ε^2 and dropped in this calculation, so that on integration we obtain

$$\frac{e}{c} \lambda_i^{(1)} + 3r \left(\frac{B_\theta f_i}{B^2} \right)^{(1)} = 0. \quad (78)$$

and

$$-\frac{e}{c} \lambda_e^{(1)} + 3r \left(\frac{B_\theta f_e}{B^2} \right)^{(1)} = 0. \quad (79)$$

In (78) and (79) we have not added an arbitrary function of flux on the right hand side, as we did in (37), (38). Any arbitrary function of flux that might appear in (78), (79) is added into the lowest order functions $F_i(\psi)$, $F_e(\psi)$ that appear in (35), (36). In general, we do not introduce any new arbitrary functions in first order; we assume that they are all absorbed in the lowest order functions. If we were to carry this process beyond first order we would have to take into account these modifications, but to first order no problems occur. If we expand out explicitly

(78) and (79), we find

$$\frac{e}{c} \lambda_i^{(1)} + 3 r^2 f_i^{(1)} / \chi_0 = I_1, \quad (80)$$

$$-\frac{e}{c} \lambda_e^{(1)} = I_2, \quad (81)$$

where I_1 and I_2 are some explicit inhomogeneous terms that depend on lowest order variables only. To obtain (81), we recall that f_e is automatically of order εp , so that only lowest order electron variables appear in f_e . The remainder of conservations of momentum is

$$\begin{aligned} \nabla p_i^{(1)} = & -e (n \nabla \varphi)^{(1)} - \frac{e}{c} \frac{\chi_0}{r^2} \nabla \lambda_i^{(1)} + \frac{e}{c} (n \omega_i \nabla \psi)^{(1)} \\ & - \nabla f_i^{(1)} - 3 \hat{r} f_i^{(1)} / r + I_3 \end{aligned} \quad (82)$$

and

$$\nabla p_e^{(1)} = e (n \nabla \varphi)^{(1)} - \frac{e}{c} (n \omega_e \nabla \psi)^{(1)} + I_4, \quad (83)$$

where we have used (81) to obtain (83), and $I_4 \neq 0$, and

$$\begin{aligned} f_i^{(1)} = & \eta_0^i \{ \chi_0 \mathbf{B} \cdot \nabla \omega_i \\ & + \chi_0^2 (\lambda_{i,r} n_{,z} - \lambda_{i,z} n_{,r}) / 3 r^3 n^2 \\ & - \lambda_{i,z} \chi_0^2 / r^4 n \}^{(1)} + I_5. \end{aligned} \quad (84)$$

Consistent with our comments we do not expand $T_i(\psi, t)$ and $T_e(\psi, t)$ in powers at ε . Maxwell's equations are essentially unchanged, and

$$\mathbf{B}^{(1)} = -\hat{r} \psi_{,z}^{(1)} / r + \theta \chi^{(1)} / r + \hat{z} \psi_{,r}^{(1)} / r, \quad (85)$$

$$\Delta^* \psi^{(1)} = -4\pi (e/c) r^2 \{ n (\omega_i - \omega_e) \}^{(1)} \quad (86)$$

and

$$\mathbf{E} = -\nabla \varphi^{(1)}. \quad (87)$$

The integral constraints of conservation of energy are (23) for ions and for electrons expanded one border beyond (44) and (45). These two equations are in the time scale τ_e / ε^2 and include the time derivatives $\partial T_i^{(1)} / \partial t$ and $\partial T_e^{(1)} / \partial t$. In addition conservation of momentum in the θ direction adds the two integral constraints (31), (38), which relate the particle flows, the momentum sources, and the electromotive forces.

It is clear that the structure of the system (71)–(87) is not substantially different from our lowest order system. We treat this system just as the

lowest order system and we start with electron momentum balance, (83). We may use (83) to determine $(n \omega_e)^{(1)}$ provided the component of (83) perpendicular to $\nabla \psi^{(0)}$ is satisfied. That is we require

$$\begin{aligned} k T_e^{(0)} \mathbf{B}^{(0)} \cdot \nabla n^{(1)} \\ = e n^{(0)} \mathbf{B}^0 \cdot \nabla \varphi^{(1)} + e n^{(1)} \mathbf{B}^0 \cdot \nabla \varphi^{(0)} + \mathbf{B}^0 \cdot \mathbf{I}_4, \end{aligned} \quad (88)$$

or

$$\begin{aligned} k T_e^{(0)} \mathbf{B}^{(0)} \cdot \nabla (n^{(1)} / n^{(0)}) \\ = e \mathbf{B}^{(0)} \cdot \nabla \varphi^{(1)} + \mathbf{B}^{(0)} \cdot \mathbf{I}_4 / n^{(0)}. \end{aligned} \quad (89)$$

Thus, we must impose the additional constraint

$$\int_{\psi \cong \bar{\psi}} \mathbf{B}^{(0)} \cdot \mathbf{I}_4 / n^{(0)} dV = 0, \quad (90)$$

and provided (90) we may determine $\varphi^{(1)}$ from (89) if $n^{(1)}$ is known. Now $\varphi^{(1)}$ is determined up to an arbitrary function of flux, and we normalize our solution, as discussed before, by the requirement that the flux surface average of $\varphi^{(1)}$ is zero. We can then determine $\varphi^{(1)}$ and $(n \omega_e)^{(1)}$ in terms of $n^{(1)}$.

The analysis of momentum balance exactly follows the pattern of the last section in the reduction from (39) and (41) to a form analogous (59) and (60) to the final forms analogous to (65), (72), (73), (74) with added inhomogeneous terms. The analysis of the hyperbolic system should then apply directly, so that solutions of ion momentum balance would exist, dependent on one free function. We can obtain a unique first order solution if we add the constraint, just as before, that $\lambda^{(1)}$ averaged over a flux surface is zero. Again, such a constraint only trivially affects the lowest order solution of the last section. Thus, in the abstract we can construct a unique solution of our first order perturbed system provided the constraint (90) is satisfied.

We may finally determine the slow time evolution of our lowest order state when we recognize that our lowest order solution depends on five free functions, see the discussion at the end of Sect. III, and we have the five integral constraints (23) for electrons and for ions, (44), (45), and (90). With this system the time derivation $\partial T_i / \partial t$, $\partial T_e / \partial t$, $\partial \psi / \partial t$, and $\partial n / \partial t$ are related and the slow time evolution is determined. It should be noted that in order to apply (44), (45) it is necessary to evaluate $\partial \psi / \partial t$. We obtain this quantity from the time derivative of the Grad-Schlüter-Shafranov equation. For this latter equation we must also differentiate the three

equations for n , λ , and ω . A similar analysis has been described in [16]. We note in passing, that with energy sources, but no momentum sources, a steady state with $\partial/\partial t = 0$ is in principle possible, provided the sources and special functions are appropriately chosen. Such a system is a steady state current driven tokamak.

V. Discussion

Before we discuss the implications and significance of the work presented here, we must comment on our use of the Braginskii model in comparison with neoclassical models. We do not assert that Braginskii transport is entirely adequate to represent tokamak physics. Rather, we believe that in view of its relative simplicity, it is a useful starting point for the exploration of various transport effects. We explain shortly why it should be a rough approximation to better models, but first we caution the reader as to the nature of the proper comparison between Braginskii and neoclassical models. Typically, in neoclassical theory one starts with a kinetic equation and by several approximations one arrives at a system of flux surface averaged equations [17]. One should not compare the coefficient derived in neoclassical theory that relate the flux surface averaged strains to the flux surface averaged forces, or stresses, to the coefficients of the local Braginskii theory. Thus, following Braginskii we take a very large coefficient of parallel viscosity, but the flux surface average of the momentum transport from parallel viscosity is identically zero in the toroidal direction and small in the direction of the magnetic field. It is these latter two quantities which are calculated in neoclassical theory. Hence our large parallel viscous coefficient is not automatically inconsistent with neoclassical ideas.

Next, we turn to the question of the approximate validity of the Braginskii model. Its usual derivation argues that enough collisions have occurred to bring the system near local thermodynamic equilibrium, and then one inserts into the Boltzmann equation a distribution function which is a local Maxwellian plus small corrections and one solves iteratively for the small corrections. There is no question as to the validity of this formulation when collisions are strong. However, if the system is for whatever

reason in local thermodynamic equilibrium – e.g. as a result of instabilities, or contact with an exterior thermodynamic equilibrium (heat bath), or method of plasma production – and if the “strains” – the gradients of equilibrium density, temperature, and velocity – are not large, then the same iterative procedure to calculate the perturbed distribution function in Braginskii can be used in our case. One will then find the “stresses” – the generalized forces generated by the “strains” as moments of the perturbed distribution function and they will be essentially the same as Braginskii. In our use of Braginskii transport we have carefully adjusted the “strains”, corresponding to parallel viscosity, for instance, to be small enough that the associated “stresses” are compatible with the other usual thermodynamic stresses, such as pressure gradient or Lorentz force. If our viscous stresses were much larger than the other forces, then the applicability of Braginskii would be questionable. Thus, we expect that the Braginskii model is a fair first approximation with which to start a study of plasma transport. We certainly acknowledge the great desirability of repeating our analysis with a full kinetic model and showing in greater detail the effects referred to in this paragraph.

We now turn to the implications of this work in the study of tokamak dynamics. We have shown that the particle flows necessary to generate equilibrium currents are large enough to have other significant effects. When one includes the effect of parallel viscosity coming from gradients of the equilibrium flows – recently other authors have considered other types of parallel viscosity – one finds that these flows also determine the energy balance and energy flow of the system. We expect this conclusion to apply more generally than in the Braginskii model. Furthermore, crude scaling arguments indicate that the energy lifetimes are in rough, qualitative agreement with present hot tokamaks. These flows allow the possibility of ideal magnetohydrodynamic equilibria, but they also allow for steady states with density varying on a flux surface, although temperature remains constant on a flux surface.

We have finally presented a model in which one calculates an equilibrium state, Sect. III, perturbations from that equilibrium state and the time evolution of the equilibrium, Section IV. The calculation of the equilibrium is considerably more

complicated than the calculation of an ideal magnetohydrodynamic equilibrium. We retain, of course, a Grad-Shafranov-Schlüter equation for the magnetic flux, but we have a set of first order differential equations for the sources in the Grad-Shafranov-Schlüter equation. This analysis indicates how intricate tokamak transport is likely to be and how sensitive the results would be to small changes. We can calculate the energy transport only after the completion of the calculation of a perturbed equilibrium calculation. Thus, it seems likely that obscure changes in lowest order may have major changes in the energy flow. We indicated that in principle our system appears to allow for a steady state tokamak with only energy sources to sustain it. Thus, we believe that we have included the “bootstrap current” of recent great interest.

We have just indicated how complicated within our model and scaling is the determination of energy transport. We could take another scaling of the velocities, for instance, in Table 2 we might multiply all the elements in the last four columns by $\varepsilon^{-1/2}$. In some ways this might fit the numbers of Table 1 slightly better. The basic model of Sect. II would not change, although the lowest order equilibrium would be rather different, and one would still get a system similar to Section III. However, the calculation of the energy transport would now require calculation of perturbed equilibria through second order. Thus, the change in the energy transport from the relatively simple results given in this paper to those with the modified scaling of velocities would be quite large. We would then expect that it is extremely unlikely that there is any

generally valid law for the scaling of energy confinement time in tokamaks with different parameters.

In the matter of “anomalous electron energy transport” we observe that in our model we have significant energy flows generated by the ion and the electron parallel viscosity. One could make the ion energy production quite small – and to the extent that a tokamak is up-down symmetric this effect must be small – but if $\beta_p \neq 1$ then there must be electron poloidal flow and correspondingly electron viscous energy production and electron energy outflow. These electron energy outflows must occur and do not require any anomalous perpendicular heat conductivity. Their effects can be estimated as on the order of ε^{-4} larger than the classical (Braginskii) perpendicular electron heat flow and ε^{-2} larger than the classical perpendicular ion heat flow. Thus, we find significant anomalous effects without the addition of anomalous transport. We believe that this phenomenon also is independent of the details of the Braginskii model and would reappear in a full kinetic calculation which includes the equilibrium flows associated with the currents. We consider that the exploration of this possibility have been the major function of this paper.

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